Chapter 14

Bifurcations in the Quadratic Map

We will approach the study of the universal period doubling route to chaos by first investigating the details of the quadratic map. This investigation suggests a pattern of behavior. Other maps with a "similar" shape, e.g. the sine map ($f(x) = \frac{1}{4}a\sin(\pi x)$), show a *quantitatively* similar behavior. We are led to a very general formulation that explains this common or "universal" behavior. From this general formulation we argue that the universality goes far beyond the iteration of maps of the unit interval, and may apply to higher dimensional maps, flows (dynamical systems described by ordinary differential equations), and physical systems. Thus *quantitative* predictions about experimental systems may be made. The detailed understanding allows the consideration of other interesting questions, such as the effect of extrinsic noise on the phenomenon of chaos.

A good reference for this material is the pedagogical article by Feigenbaum [1], who is responsible for many of the ideas.

14.1 Patterns in the quadratic map

For the quadratic map $f(x) = ax(1-x)$ the "bifurcation diagram" and plot of Lyapunov exponent λ as a function of the map parameter a shows that an infinite sequence of transitions or bifurcations occurs as a is increased between 3.0 and about 3.57, and there are patterns to be seen in the structure of the transitions (see chapter 3). The bifurcations are of successive period doubling, i.e. the orbit (after transients have died out) for small a is a single point; but as a is increased the orbit consists of 2 points, then $4, 8 \ldots 2^n, \ldots$ points. At each bifurcation the period doubles, and the power spectrum shows successive subharmonic peaks (at frequencies $\frac{1}{2}, \frac{1}{4}, \ldots$ relative to the iteration frequency).

We would like to look at the *quantitative* structure of this sequence of bifurcations. We could look at the pattern in the bifurcation points themselves, i.e. the a_n for the bifurcation from the 2^n cycle to the 2^{n+1} cycle. However the dynamics becomes slow near a bifurcation point (the Lyapunov exponent goes to zero), and so transients take longer to die out. Instead we will look at a particular value $a_n^{(s)}$ within each 2^n cycle. A convenient choice of the particular value is the value of a giving the "superstable" 2^n cycle. This is the value for which one point of the orbit is exactly at the maximum of the map. At this value the Lyapunov exponent is (negative) infinity—hence the name. These values of a are easily identified from the $\lambda(a)$ plot, and also from the bifurcation plot where the line of an orbital point crosses $x = \frac{1}{2}$, and transients decay rapidly here.

In addition to the values $a_n^{(s)}$, we can ask for the separation of points in the orbit. At each bifurcation, every point in the orbit splits into two points: we are interested in how this small splitting develops as n increases. The closest point to any given one is the point half way around the orbit (which was coincident prior to the bifurcation). For the superstable cycles the smallest separation between any two points in the orbit is the separation between the point at the maximum and the point half way around the cycle from this point. Again this is easily identified from the bifurcation map or extracted numerically (start at $x = \frac{1}{2}$ and iterate 2^{n-1} times for $a = a_n^{(s)}$.)(demonstration 1)

The following results are readily extracted for the quadratic map:

n	Period	$a_n^{(s)}$	δ_n	d_n	α_n
0	1	2		0	
1	2	3.236065	1.61804	0.309016	
2	4	3.498562	4.70887	−0.116402	−2.65474
3	8	3.554641	4.68083	0.045975	−2.53184
4	16	3.566667	4.66294	−0.018326	−2.50872
5	32	3.569244	4.66840	0.007318	−2.50411
6	64	3.569795	4.66895	−0.002924	−2.50316
7	128	3.569913	4.66917	0.001168	−2.50296
8	256	3.569939	4.66919	−0.000467	−2.50292
9	512	3.569944	4.66920	0.000186	−2.50291

The separation between the $a_n^{(s)}$ decreases rapidly, and in fact geometrically, as is shown by calculating the ratio between successive separations, or its inverse

$$\delta_n = \frac{a_{n-1}^{(s)} - a_{n-2}^{(s)}}{a_n^{(s)} - a_{n-1}^{(s)}}. \tag{14.1}$$

The values of δ_n are tabulated in the third column. (Note these are calculated from the values of $a_n^{(s)}$ in double precision before truncation to fit in the table!) It is apparent that δ_n tends to a constant for large n, in fact

$$\lim_{n\to\infty} \delta_n \to \delta = 4.6692016\ldots \tag{14.2}$$

Similarly the smallest separation d_n between points in the 2^n orbit decreases geometrically, as shown by the tabulation of

$$\alpha_n = \frac{d_{n-1}}{d_n}. \tag{14.3}$$

Again this ratio approaches a constant, in fact given by

$$\lim_{n\to\infty} \alpha_n \to \alpha = -2.502907875\ldots \tag{14.4}$$

We can write the relationship (14.2) as

$$a_n^{(s)} \to a_\infty - A^{(s)}\delta^{-n} \tag{14.5}$$

where for the quadratic map $a_\infty = 3.569946\ldots$ and $A^{(s)} \simeq 1.5561$. The bifurcation points follow a similar expression

$$a_n^{(b)} \to a_\infty - A^{(b)}\delta^{-n} \tag{14.6}$$

with $A^{(b)} \simeq 0.570$.

14.2 Patterns in the sine map

The sine map $f(x) = \frac{1}{4}a\sin(\pi x)$ shows a similar looking sequence of bifurcation demonstration 2. We can make a similar table:

n	Period	$a_n^{(s)}$	δ_n	d_n	α_n
0	1	2		0	
1	2	3.110931	1.80031	0.2777733	
2	4	3.385529	4.04565	−0.107204	−2.59069
3	8	3.445801	4.55592	0.042518	−2.52139
4	16	3.458777	4.64516	−0.016962	−2.50668
5	32	3.461559	4.66407	0.006775	−2.50370
6	64	3.462155	4.66813	−0.002707	−2.50308
7	128	3.462282	4.66895	0.001081	−2.50294
8	256	3.462310	4.66915	−0.000432	−2.50291
9	512	3.462315	4.66920	0.000173	−2.50291

The specific numbers $a_n^{(s)}$ and d_n are *different*, however we notice the remarkable result that the parameters δ_n and α_n appear to tend towards the *same* values as $n \to \infty$! Again we can write (14.5) and (14.6) with *the same value of* δ but with $a_\infty = 3.46231\ldots$, $A^{(s)} \simeq 1.6821$, and $A^{(b)} \simeq 0.614$.

This result can be extended to various other one dimensional maps with the common feature of a quadratic maximum. Thus there are factors that set the overall scale of the variation with a and the orbit size, but the ratios of separations tend to universal values for large n 2^n cycles. This is the remarkable universality discovered by Feigenbaum. It covers a large class of maps with a quadratic maximum (there is also a requirement of "positive Schwarzian derivative"). However it can be readily seen (demonstration 3) that maps that vary with a different power law near the "maximum" (which is now actually a cusp) show *qualitatively* similar geometric convergences at large n that however are characterized by *different* values of δ and α.

14.3 Period Doubling Bifurcations

Before developing further understanding of the geometric sequence of period doubling bifurcations it is useful to introduce a few basic properties of bifurcations in the quadratic map. We will discuss properties of a map $f(x)$, which may be the quadratic map, or the maps given by "functional composition" that give higher order iterates of the point. For example

$$f^2(x) \equiv f(f(x)) \tag{14.7}$$

gives the second iterate of the map function f

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) \end{aligned} \quad . \tag{14.8}$$

Note again that f^2 denotes the (second order) functional composition, not the square of the function f. Similarly

$$f^n(x) = f(f^{n-1}(x)) = f(f(f \dots f(x))) \tag{14.9}$$

gives the nth iterate of x.

14.3.1 Useful properties

1. A fixed point of x_f is given by

$$x_f = f(x_f). \tag{14.10}$$

2. The stability of the fixed point is determined by perturbing about x_f, i.e. write $x_n = x_f + \delta x_n$ with δx_n small, then

$$\delta x_{n+1} = f(x_f + \delta x_n) - x_f = f'(x_f)\delta x_n. \tag{14.11}$$

   Thus we have stability (δx_n decays) for $\left|f'(x_f)\right| < 1$, whereas we have instability (δx_n grows) for $\left|f'(x_f)\right| > 1$. The period doubling bifurcation corresponds to the borderline with $f'(x_f) = -1$.

3. The chain rule of differentiation gives

$$\left.\frac{d}{dx} f^n(x)\right|_{x=x_0} = f'(x_{n-1}) f'(x_{n-2}) \dots f'(x_1) f(x_0) \tag{14.12}$$

   where x_i are the iterates of x_0. This is shown by the result

$$\frac{d}{dx} f^n(x) = \frac{d}{dx} f(f^{n-1}(x)) = f'(f^{n-1}(x)) \frac{d}{dx} f^{n-1}(x). \tag{14.13}$$

4. A fixed point of f is a fixed point of f^n, since

$$f^2(x_f) = f(f(x_f)) = f(x_f) = x_f \tag{14.14}$$

   etc.

5. If a fixed point of f becomes unstable then it is unstable in f^n, since from (14.12)

$$\left|f^n(x_f)\right| = \left|f(x_f)\right|^n \tag{14.15}$$

   since all the x_i in (14.12) are equal to x_f.

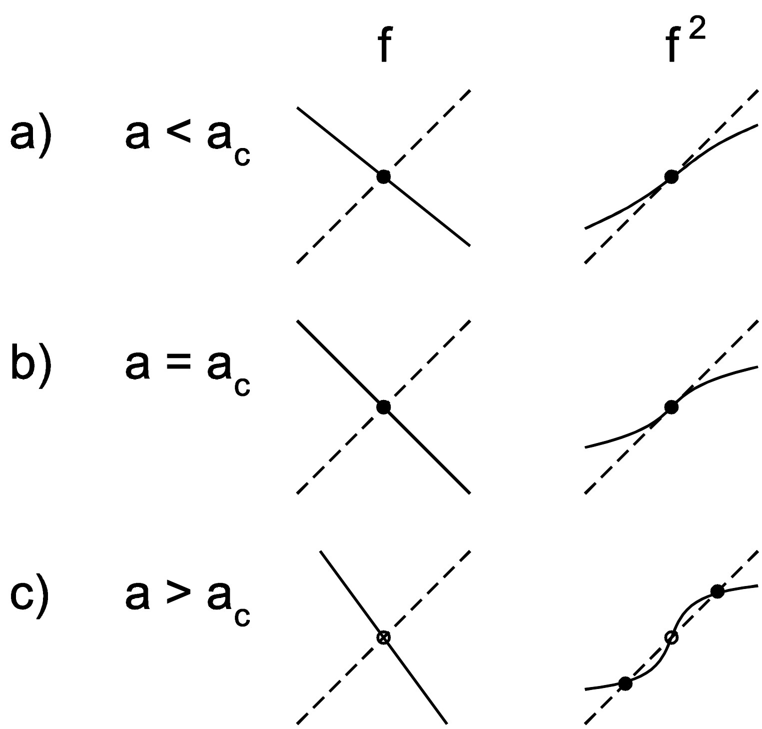

Figure 14.1: Plot of $f(x)$ and $f^2(x)$ near the fixed point x_f (intersection with $y = x$). Solid circles denote stable fixed points; empty circles unstable fixes points.

6. The structure of a subharmonic (period doubling) bifurcation is shown by figure 14.1. As a increases past the instability point of x_f the fixed point of f, the point x_f becomes unstable. The slope of f is near -1, and the slope of f^2 is then near 1. The point x_f is also a fixed point of f^2 and becomes unstable at the same value of a (panel b). Above the instability point the slope of f^2 at x_f is greater than unity. To maintain the continuity of the curve f^2 we must have two new fixed points x_1, x_2 that grow out of x_f, and at these two points the slope of f^2 is less than unity, i.e. they are *stable* fixed points of f^2. These two points are *not* fixed points of f, but since $f^2(x_1) = f(f(x_1))$ we see that

$$\begin{aligned} x_2 &= f(x_1) \\ x_1 &= f(x_2) \end{aligned} \tag{14.16}$$

   i.e. x_1 and x_2 are the points of a period 2 orbit of f.

7. On increasing a the two fixed points of f^2 become unstable together, since

$$\left[f^2(x_1)\right]' = f'(x_1)f'(x_2) = \left[f^2(x_2)\right]'. \tag{14.17}$$

8. If f has a maximum at $x = \frac{1}{2}$, then for $f(\frac{1}{2}) > \frac{1}{2}$ the function f^2 has 3 stationary points, at $x = \frac{1}{2}$, and at the two preimages of $x = \frac{1}{2}$ (i.e. the two x_p such that $f(x_p) = \frac{1}{2}$. This follows from (14.12).

Bibliography

[1] M. Feigenbaum, Los Alamos Science **1**, 4 (1980). Reprinted in “Universality in Chaos” by P. Cvitanović (IOP Publishing, Bristol 1989).