Chapter 19

Circle Map

The one dimensional circle map

$$x_{n+1} = x_n + \Omega - \tfrac{K}{2\pi} \sin 2\pi x_n, \tag{19.1}$$

where we can collapse values of x to the range $0 \le x \le 1$, is sufficiently simple that much is known rigorously [1], yet displays the interplay between frequency locking and the onset of chaos from quasiperiodic motion. There are also a number of excellent reviews e.g. references [2] and [3]. The map is monotonic and invertible for $K < 1$, but develops an inflection point at $x = 0$ for $K = 1$ and then is nonmonotonic and noninvertible for $K > 1$ (demonstration 1).

An important quantity is the winding number W defined as

$$W = \lim_{n\to\infty} \frac{x_n - x_0}{n}. \tag{19.2}$$

It is the average frequency of the motion around the circle. It is important to note that in defining W we take x_n defined by the iteration of (19.1), i.e. before we use "mod 1" to give a value in the unit interval. In fact if we write $\bar{x}_n = x_n \bmod 1$ then W is just the limit of $(x_n - \bar{x}_n)/n$ for large n.

The properties of the solution are shown in figure 19.1.

19.1 Properties for $K < 1$

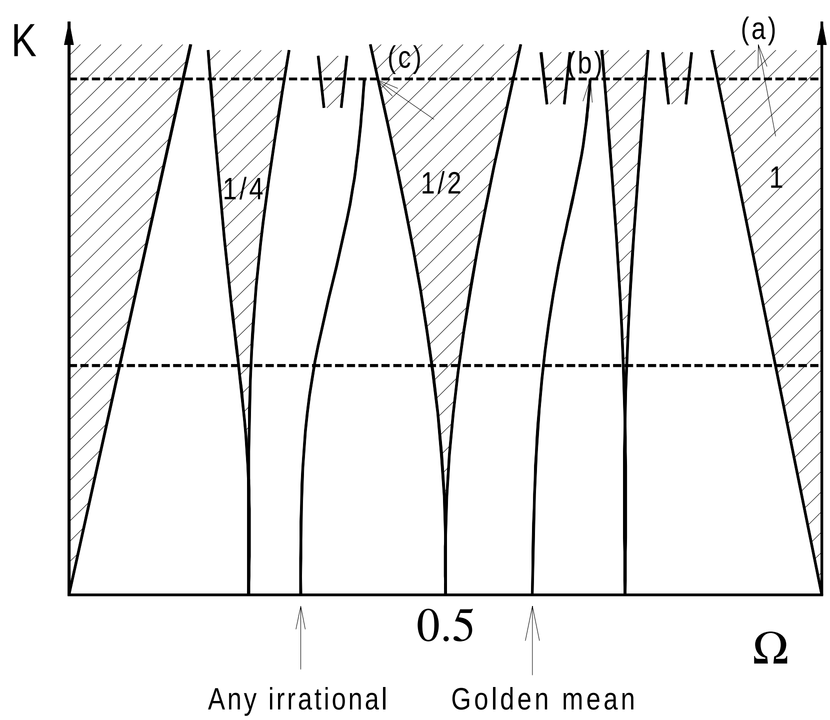

Figure 19.1: Solution structure of the circle map

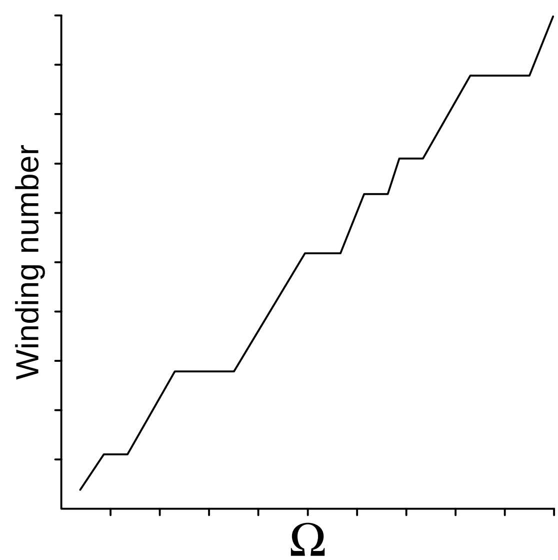

Figure 19.2: A poor representation of the "devil's staircase" function of the winding number W as a function of the linear frequency Ω. Actually each portion of riser (slanting lines) would look something like the whole sketch, since there are steps at *every* rational W.

1. Since the map is smooth and monotonic, if we start from a sequence of initial conditions, the order of points in the sequence is not changed by iteration, and this can be used to show that there is no chaos. The motion is periodic, and characterized by a winding number W that may be either *rational* corresponding to locked motion (to the strobe frequency) or *irrational* corresponding to unlocked motion, i.e. quasiperiodic motion in the flow (demonstration 2).

2. The winding number W is a continuous function of Ω for fixed $K < 1$ (e.g. along the dotted line in figure 19.1), although certainly not smooth.

3. $W(\Omega)$ passes through each irrational and has a step at each rational p/q (p, q integers), i.e. there is a range of Ω giving a constant $W = p/q$, i.e. the plot of $W(\Omega)$ is a staircase with an infinite number of tiny steps, joined by smooth risers. Such a function is known as a "devil's staircase". It is very hard to draw! A poor attempt is shown in figure 19.2.

4. The sum of the widths of the rational steps (i.e. the total length of the treads) tends to zero as $K \to 0$ and to unity as $K \to 1$ (but nevertheless each irrational is also represented for $K \to 1$). The measure (in Ω) of rational and irrational W are both nonzero for $0 < K < 1$.

5. The regions of locked motion, growing from rational Ω on the $K = 0$ line are known as Arnold tongues.

6. The size of the rational steps can be ordered depending on the size of the denominator q: very roughly, larger q is "closer" to an irrational and gives a smaller step. The full structure of the ordering of the step sizes can be worked out in detail, and is known as the Farey Tree, see e.g. [2].

7. Any irrational winding number gives a continuous curve $\Omega = \Omega_W(K)$ from $\Omega_W(0) = W$ all the way to $K = 1$.

19.2 Properties for $K = 1$

1. At $K = 1$ the map develops an inflection point at $x = 0$.

2. The measure of Ω giving irrational W is zero. The set of Ω giving irrational W is a fractal of dimension (found numerically) 0.87.

3. The periodic solutions with irrational winding number experience all points of the map, including the inflection point, and break down to chaos at $K = 1$ (path (b) in figure 19.1, demonstration 3).

4. The periodic solutions with rational winding numbers only experience discrete points of the map, typically away from the inflection point, and so typically do not break down to chaos at $K = 1$. However we can follow a path in the (K, Ω) plane leaving a periodic tongue just at $K = 1$ (path (c) in figure 19.1) when again chaos will develop at $K = 1$.

19.3 Properties for $K > 1$

1. The periodic tongues may overlap, giving hysteresis.

2. The map now has a quadratic maximum, and the periodic solutions may break down to chaos through period doubling cascades (demonstration 4).

Bibliography

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[3] P. Bak, T. Bohr, and M.H. Jensen, Phys. Scr. **T9**, 50 (1985)