Chapter 24

Controlling Chaos

Chaotic attractors contain unstable periodic orbits of any desired period (this is shown in chapter 22 for Axiom A attractors). Furthermore, for an ergodic attractor we know that any trajectory will eventually come arbitrarily close to any of these orbits. This offers the opportunity for *controlling chaos*: when the chaotic orbit approaches the unstable periodic orbit of interest it can be attracted to and maintained on the orbit by applying small perturbations to the system. There are two aspects to the problem. The first, depending on the properties of the whole attractor, is the idea that waiting long enough guarantees that the orbit will come arbitrarily close to any chosen point on the attractor. Alternatively knowledge of the chaotic attractor can be used to speed this process by directing the orbit to the desired region—this idea is studied further in the next chapter. The second part is to use delicate perturbations of the system to keep the orbit on or very close to the unstable orbit. This part can be analyzed in terms of small perturbations from the orbit, i.e. by a *linear* analysis. There is an enormous literature in "Control Theory" in engineering that can be taken over for this second part. An intriguing aspect of control in the context of chaotic systems is that *different* periods can be selected simply by changing the nature of the delicate perturbations of the system.

The idea of "controlling chaos" was suggested in a famous paper by Ott, Grebogi and Yorke [1], and we will first study the idea in the context they used of chaos in a two dimensional map (which might be the Poincaré section of a three dimensional flow). Two reviews are [2] and [3].

24.1 The OGY Method

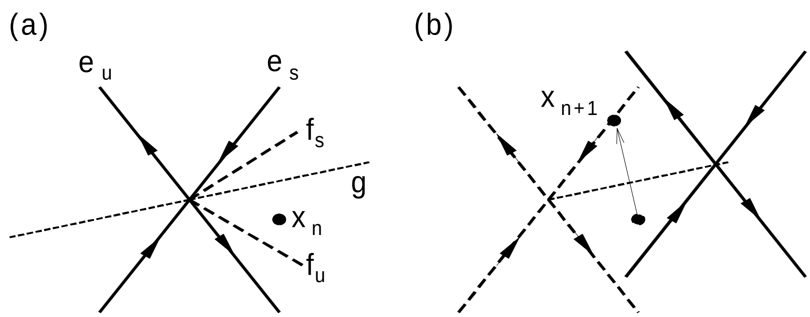

Figure 24.1: The OGY approach to control. (a) The unstable fixed point, stable and unstable eigenvectors, e_s and e_u, and their adjoints, f_s and f_u. The dotted line is the path of the fixed point as the control parameter is varied. (b) Shifting the fixed point to the new (solid) position allows x_{n+1} to be directed to the stable manifold of the unshifted (dashed) fixed point.

Suppose we wish to control the chaotic dynamics of a two dimensional map F onto an unstable fixed point $\vec{x}_f$. (Controlling to an unstable period-n orbit can be achieved by considering the corresponding fixed point of the map $\bar{F} = F^n$, although this might not be the most efficient way in practice). We suppose the fixed point has one stable direction $\vec{e}_s$ with eigenvalue λ_s and one unstable direction $\vec{e}_u$ with eigenvalue λ_u, as would be typical for an attractor of dimension less than 2. The goal is to achieve control through small changes to a parameter p of the map. The OGY scheme for control is easily understood pictorially. Figure (24.1) depicts the unstable fixed point $\vec{x}_f$ with its stable and unstable eigenvectors, and also the path of the fixed point for small changes in p given by

$$\vec{g} = \frac{\partial \vec{x}_f}{\partial p}. \tag{24.1}$$

If an iteration point $\vec{x}_n$ comes close to $\vec{x}_f$ i.e. $\vec{x}_n = \vec{x}_f + \delta\vec{x}_n$ with $\delta\vec{x}_n$ small, the parameter p is changed $p = p_0 + \delta p_n$ so that after the next iteration $\vec{x}_{n+1}$ lies along the stable direction of $\vec{x}_f$. We can calculate $\vec{x}_{n+1}$ by linearizing about the moved position of the fixed point

$$(\delta\vec{x}_{n+1} - \delta p_n\, \vec{g}) = \left(\lambda_u \vec{e}_u \vec{f}_u + \lambda_s \vec{e}_s \vec{f}_s\right) \cdot (\delta\vec{x}_n - \delta p_n\, \vec{g}) \tag{24.2}$$

where $\vec{f}_u$ and $\vec{f}_s$ are the unit adjoint eigenvectors

$$\vec{f}_u \cdot \vec{e}_s = \vec{f}_s \cdot \vec{e}_u = 0 \tag{24.3}$$

so that the resolving a vector $\vec{v}$ along $\vec{e}_u$ and $\vec{e}_s$ gives components $\vec{v} \cdot \vec{f}_u$ and $\vec{v} \cdot \vec{f}_s$ respectively. The condition for $\vec{x}_{n+1}$ to lie along the stable direction of $\vec{x}_f$ is then $\delta\vec{x}_{n+1} \cdot \vec{f}_u = 0$ which gives

$$\delta p_n = \frac{\lambda_u}{\lambda_u - 1} \frac{\delta\vec{x}_n \cdot \vec{f}_u}{\vec{g} \cdot \vec{f}_u}. \tag{24.4}$$

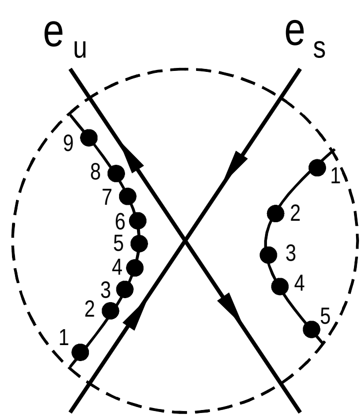

Figure 24.2: Two trajectories near the unstable fixed point. The values x_j are used to determine the position of the fixed point and the Jacobean A.

This is trivial to implement if we have analytic knowledge of the map function F. Usually we are interested in *empirical control* where the parameters have to be extracted from observations of the dynamics. This is done in three stages:

1. Identify the unstable periodic orbit(s) or order n: we need to find an $\vec{x}$ such that $F^n(\vec{x})$ is sufficiently close to $\vec{x}$. Thus we form a vector $\vec{x}_i, \vec{x}_{i+1} \ldots \vec{x}_{i+n}$ and test whether $\vec{x}_{i+n}$ is equal to $\vec{x}_i$ within some chosen tolerance, but not equal to any intermediate $\vec{x}_{i+j}$ (within the tolerance) since this would indicate a lower period orbit. If the test fails increment i and try again.

2. Characterize the unstable fixed point of $\bar{F} = F^n$: every time $\vec{x}_n$ comes close to $\vec{x}_f$ (within some tolerance) successive returns are fit to the linear form

$$\vec{x}_{j+1} - \vec{x}_f = A\left(\vec{x}_j - \vec{x}_f\right) \tag{24.5}$$

until $\vec{x}_j$ moves too far away from $\vec{x}_f$, with A the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{24.6}$$

where $\vec{x}_f$ and the coefficients a_{ij} are determined by a least squares fit (figure 24.2). The matrix A is then diagonalized to give the eigenvalues λ_u, λ_s, the eigenvectors $\vec{e}_u$, $\vec{e}_s$, and the adjoint eigenvectors $\vec{f}_u$, $\vec{f}_s$. The vector $\vec{g}$ is found by slightly incrementing the parameter p and extracting the new $\vec{x}_f$ by a similar least squares fit.

3. Control is now implemented with these estimated parameters using (24.4). Since the control is estimated using linearization, it is only implemented when $\vec{x}_n$ comes sufficiently close to $\vec{x}_f$.

A slight wrinkle to this method is added if the observed chaotic dynamics is reconstructed from the measurements of a single dynamical variable [4]. For the case of a two dimensional map the relationship between the reconstructed map and the phase space map depends on the map parameter at the previous iteration

$$\vec{x}_{n+1} = F\left(\vec{x}_n; p_n, p_{n-1}\right) \tag{24.7}$$

so that now

$$\left(\delta\vec{x}_{n+1} - \delta p_n\,\vec{g} - \delta p_{n-1}\,\vec{h}\right) = \left(\lambda_u\vec{e}_u\,\vec{f}_u + \lambda_s\vec{e}_s\,\vec{f}_s\right)\cdot\left(\delta\vec{x}_n - \delta p_n\,\vec{g} - \delta p_{n-1}\,\vec{h}\right) \tag{24.8}$$

with $\vec{g} = \partial\vec{x}_f/\partial p_n$ and $\vec{h} = \partial\vec{x}_f/\partial p_{n-1}$. This leads to

$$\delta p_n = \frac{\lambda_u}{\lambda_u - 1}\,\frac{\delta\vec{x}_n\cdot\vec{f}_u}{\vec{g}\cdot\vec{f}_u} - \frac{\vec{h}\cdot\vec{f}_u}{\vec{g}\cdot\vec{f}_u}\delta p_{n-1}. \tag{24.9}$$

The new vector $\vec{h}$ can be "learned" by applying small changes δp_n to p that are on for n even and off for n odd.

The OGY control scheme is illustrated in the demonstration.

24.2 General Linear Control Theory

The OGY control method is very appealing intuitively. However it is only one of a large class of possible algorithms. In addition, it is very easy to guess incorrect generalizations from this (perfectly correct) special case. For example it might be guessed that a new control parameter is needed for each new unstable direction at the fixed point in higher dimensional situations. This is *not* the case: typically only a *single* control parameter is sufficient even when multiple directions are unstable. Since the issue is *linear* control near the fixed point or periodic orbit, the fact that we are dealing with a *chaotic* system is of secondary importance in the control algorithm, and standard results from linear control theory [5],[6] can be taken over directly.

We will consider the framework of more general control methods for unstable fixed points in systems of arbitrary dimension N using a vector of M control parameters p. Linearizing near the fixed point we have

$$x_{n+1} = Ax_n + Bu_n \tag{24.10}$$

where we are measuring the dynamical variable x from the fixed point, and $u_n = p_n - p$ i.e. the perturbations of the map parameters. The matrices A and B are given by derivatives at the fixed point

$$A_{ij} = \frac{\partial F_i}{\partial x^{(j)}} \quad \text{and} \quad B_{ik} = \frac{\partial F_i}{\partial p^{(k)}}. \tag{24.11}$$

Notice that the matrix B is fixed by the choice of control parameters. (The dimension of the parameter vector need not of course be equal to the dimension of the phase space vector x, so B, an $N \times M$ matrix, need not be square.) In general we could take u_n to depend on a number of prior values of x (i.e. $x_n, x_{n-1} \ldots x_{n-m}$). However we will restrict ourselves to “proportional control”

$$u_n = -Kx_n, \tag{24.12}$$

with K and $M \times N$ matrix. Thus we have, within the linear approximation

$$x_{n+1} = (A - BK)\, x_n \tag{24.13}$$

and the properties of this linear system tells us about the possibility of control.

A system is said to be *stabilizable* if with the choice of control parameters p a “feedback gain matrix” K can be found such that all the eigenvalues of $A - BK$

lie within the unit circle i.e. $|\lambda_i| < 1$. Clearly then, with the control on, the perturbation x from the fixed point dies away exponentially.

A more strict notion is that of controllability. A system is said to be linearly *controllable* if for any initial condition x_0 close to the fixed point at $x = 0$, there exists a sequence of perturbations $u_0 \ldots u_{t-1}$ for any $t \geq N$ such that $x_t = 0$. Within the proportional scheme this is equivalent to the requirement that each eigenvalue of the matrix $A - BK$ can be chosen at will, and in particular can be made zero.

Using linear algebra, the condition for controllability can be easily constructed. The explicit expression for x_t for $t = N$ is

$$x_N = A^N x_0 + \sum_{j=0}^{N-1} A^{N-1-j} B u_j . \tag{24.14}$$

Denote the ith column in the matrix B as b_i:

$$B = [b_1 : b_2 : b_3 : \ldots : b_M] . \tag{24.15}$$

Regarding the terms

$$f_i^j = A^{N-j} b_i , \quad j = 1, N; \quad i = 1, M \tag{24.16}$$

as $N \times M$ basis vectors and $u_{j-1}^{(k)}$ ($k = 1, M$ and $j = 1, N$) as coordinates, we see that the condition $x_N = 0$ can only be satisfied for general x_0 if this set of vectors is complete, i.e. there are N linearly independent vectors. This is equivalent to requiring

$$rank(C) = N \tag{24.17}$$

where C is the "controllability matrix"

$$C = [B : AB : A^2 B : \quad \cdots \quad : A^{N-1} B] \tag{24.18}$$

i.e. C is the matrix with columns $[b_1 : b_2 : b_3 \ldots b_M : (Ab_1) : (Ab_2) : (Ab_3) \ldots (Ab_M) \ldots]$. For the case of proportional control this is in fact the same condition that a matrix K can be found such that the matrix $A - BK$ has any desired eigenvalues. For the single control parameter case B is a column vector b_1, and for a matrix A with nondegenerate eigenvalues the controllability condition is simply that b_1 has components along all the eigenvectors of A. If this is satisfied, the system is controllable by the *single* parameter, no matter how many of the eigenvectors correspond to unstable directions. In this one parameter case, if the controllability condition is satisfied, the matrix K can be obtained from Ackermann's formula

$$K = [0, 0, 0, \ldots 1]C^{-1}\phi(A) \tag{24.19}$$

with

$$\phi(A) = (A - \mu_1 I)\,(A - \mu_2 I)\ldots(A - \mu_n I) \tag{24.20}$$

where μ_i are the desired eigenvalues. The more complicated multiparameter case is discussed by [6].

The stabilizability condition can be written in similar form: if there are n_s stable directions with eigenvectors e_s and n_u unstable directions, the system is stabilizable if

$$rank(S) = N \tag{24.21}$$

with S the "stabilizability matrix"

$$S = [e_1 : \ldots e_{n_s} : B : AB : A^2 B : \quad \cdots \quad : A^{n_u - 1} B]. \tag{24.22}$$

The OGY method for a single unstable direction corresponds to setting the eigenvalue of $A - BK$ along the unstable direction of A to zero to give immediate convergence to the stable manifold. The "direct targeting" method aims at arriving at the fixed point after N steps, which corresponds to setting all the eigenvalues of $A - BK$ to zero. Note, of course, that in this case $A - BK$ is a degenerate matrix, and will not have N independent eigenvectors (when we would get convergence to the fixed point in a single step!).

24.3 Linear Quadratic Control

There is clearly considerable flexibility in choosing a control scheme—it is too easy to control systems! Often the method (e.g. OGY) is chosen for conceptual simplicity or by ingenuity. It would be useful to have some sort of quantitative measure of the "goodness" of any control scheme. Various measures of the quality could be imagined: the smallest number of steps; the validity over the widest deviation from the fixed point (the linear approach will break down somewhere); or the robustness of the control in the presence of noise. An attractive scheme is to minimize a cost function, for example the quadratic form in the deviations x_n and the control strengths u_n

$$V(x_0) = \sum_{n=1}^{\infty} x_n^T Q x_n + u_n^T R u_n \tag{24.23}$$

where Q, R are positive symmetric matrices that are chosen to weight the different directions, and the relative importance of restricting the deviations x_n and control strengths u_n to small values (e.g. for the two linearizations (24.10) to be a useful approximation). Equation (24.23) can be minimized, subject to the constraint of the dynamics (24.10), using standard Lagrange multiplier or other methods. After considerable effort it can be shown [7] that the minimum is reached for K given by

$$K = \left(R + B^T P B\right)^{-1} B^T P A \tag{24.24}$$

where P is the symmetric matrix that is the solution to the "discrete time algebraic Ricatti equation"

$$P = Q + A^T P A - A^T P B \left(R + B^T P B\right)^{-1} B^T P A. \tag{24.25}$$

(This equation can be solved iteratively

$$P_{n+1} = Q + A^T P_n A - A^T P_n B \left(R + B^T P_n B\right)^{-1} B^T P_n A \tag{24.26}$$

which usually converges rapidly $P_n \to P$ from a typical choice of symmetric P_0.)

To get some insight into the minimization consider the situation where $A - BK$ has nondegenerate eigenvalues λ_i. We can then write

$$V(x_0) = \sum_i \frac{x_0^{(i)*} \left(Q + K^T R K\right)_{ij} x_0^{(j)}}{1 - \lambda_i^* \lambda_j} \tag{24.27}$$

with the indices i, j referring to components along the eigenvectors of $A - BK$. If we suppose x_0 is chosen at random, and average over all possible directions we want to minimize the average

$$\langle V \rangle = \sum_i \frac{\left(Q + K^T R K\right)_{ii}}{1 - |\lambda_i|^2}. \tag{24.28}$$

This does not solve the general problem, since the unknown matrix K appears, as well as the unknown eigenvalues and eigenvectors of $A - BK$. For the special case $R = 0$ and $Q = I$ (minimize the mean square distance of x_n from the fixed point) the sum reduces to

$$\langle V \rangle = \sum_i \left(1 - |\lambda_i|^2\right)^{-1} \tag{24.29}$$

which suggests the minimization is given by setting all $\lambda_i = 0$, i.e. the direct targeting algorithm (although then the eigenvalues are degenerate so the procedure may not be consistent).

An interesting result is for controlling dynamics in the presence of small additive uncorrelated noise, i.e. given by

$$x_{n+1} = F(x_n) + \xi_n \tag{24.30}$$

with ξ_n a random variable. It can be shown that minimizing $\langle V \rangle$ (24.23) in the presence of noise leads to the *same* optimization condition (24.24),(24.25).

24.4 Applications

There has been an enormous number of papers written on applications of the control of chaos. Not all papers with this phrase in the title describe schemes that fall into the framework I have discussed, namely

- control by application of small feeback signals, that go to zero or to very small values controlled by stochastic noise (not deterministic, chaotic "noise") once control has been achieved;
- control to a pre-existing unstable fixed point or periodic orbit within the attractor;
- control making intelligent use of the the structure of the dynamics near the unstable orbits.

Some schemes, using large applied signals, are more reminiscent of locking of large amplitude oscillators. Of course, schemes not implemeting all these features may still be useful! For example one early application was to controlling chaos in heart muscle [8]. Here, if the spontaneous heart beat is delayed from the expected period, a *finite* electrical stimulus is provided to force a heart beat, so the analysis in terms of a small change in the control parameter is not valid. Nevertheless, the timing of the stimulated pulse *is* determined from an OGY type analysis of the pulse time return map—simply stimulating at the expected period did *not* lead to a periodic response.

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